Problems to Explore

Rutgers Young Scholars Program

Summer 2018

PROBLEM 1: TAXES

Charles' income is 20% larger than Benjamin's income. After making \$4000 in deductions from his total income, Charles pays 12% of the remainder as income taxes. Similarly, after making \$2000 in deductions from his total income, Benjamin pays 10% of the remainder as income taxes.

However, Charles' taxes are 40% greater than Benjamin's. How much did Charles and Benjamin make in income?

PROBLEM 2: HOW MANY SCREWS?

Two carpenters visited a hardware store. The first carpenter bought an equal number of 3, 4, and 5 cent screws. The second carpenter spent the same amount of money as the first carpenter, but spent an equal amount of money on each of the 3, 4, and 5 cent screws.

If the second carpenter got 6 more screws than the first carpenter, how many screws of each type did the two carpenters buy?

PROBLEM 3: THE BRIDGE OVER THE RIVER

A bridge over a river is in the shape of the arc of a circle with each base of the bridge at the river's edge.

At the center of the river the bridge is 10 feet above the water. At 27 feet from the edge of the river, the bridge is 9 feet above the water. How wide is the river?

PROBLEM 4: ADDITIONS WITH ALL DIGITS USED EXACTLY ONCE

In the following the set $\{a, b, c, d, e, f, g, h, i, j\}$ is exactly the set of digits $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

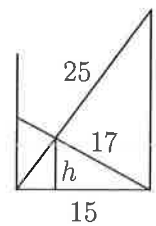
(a) Here, $a, bc, def, ghij$ are respectively one, two, three, and four digit numbers. As usual, the leading digits are not zero, i.e., $a \neq 0$, $b \neq 0$, $d \neq 0$, $g \neq 0$. The problem is to find all of the additions of the following type. Once a solution is found performing any permutation of $\{a, c, f\}$ or $\{b, e\}$ gives another solution. Thus, to avoid redundancy suppose that $a < c < f$ and $b < e$.

$$\begin{array}{r} a \\ b\ c \\ \underline{d\ e\ f} \\ g\ h\ i\ j \end{array}$$

(b) Here, $abc, def, ghij$ are respectively two three digit numbers and a four digit number. Again as before, the leading digits a, d, and g are not zero. The problem is to find all possible additions. Assume, as before that $a < d$, $b < e$, and $c < f$.

$$\begin{array}{r} a\ b\ c \\ \underline{d\ e\ f} \\ g\ h\ i\ j \end{array}$$

PROBLEM 5: CROSSED LADDERS



A room is 15 feet wide. Ladders of lengths 25 feet and 17 feet rest against opposite walls of the room, as in the picture at the right.

What is the distance h above the floor where the two ladders cross?

PROBLEM 6: FIND THE DIGITS

The integer $3^{(3^3)} = 3^{27} = 7625597484987$ has 13 digits. Its first 3 digits are 762 and its last 3 are 987. Let $x = 7^{(7^7)} = 7^{823543}$.

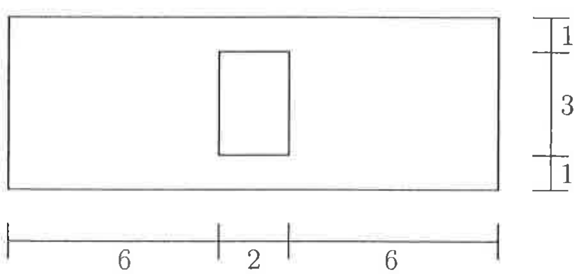
(a) How many digits does x have?
(b) What are the first 3 digits of x?
(c) What are the last 3 digits of x?

PROBLEM 7: ATTACKING QUEENS

Recall that in chess a queen attacks any square that is on a straight line — horizontally, vertically, or diagonally — from the square on which the queen stands. Regard the queen as also attacking the square which she occupies. What is the smallest number of queens required to attack all of the squares of a 6×6 chessboard?

PROBLEM 8: CARPET CUTTING

A 14×5 carpet runner has a 2×3 hole in its center. Show how to cut it into two pieces that can be re-assembled to give a 16×4 runner.



PROBLEM 9: EASY MULTIPLICATIONS

The number $x = 142857$ is easy to multiply by 3, since $3x = 428571$ is found by moving the leading digit of 142857 to the rear to get 428571.

Find the smallest positive integer x so that $(3/2)x$ can be found by moving the leading digit of x to the rear, as in the example above.

PROBLEM 10: INTEGERS ARRANGED IN A CIRCLE

If n is a non-zero digit, there are n^2 two digit integers ab that can be formed with the digits from 1 to n. If $n = 3$, these two digit integers can be arranged in a circle, so that, going around the circle in a clockwise direction, each integer ends with the digit that the following integer begins with. For example, here are two such arrangements. Starting at the top of the first circle and going around, the integers are 11, 12, 23, 32, 22, etc., and the final digit of each integer is the initial digit of the next.

	11					11	
31		12			31		12
33		23			23		21
13		32			22		13
21		22			32		33

(a) Find a similar circular arrangement for the the n^2 two digit integers that can be formed when $n = 4$ and $n = 5$.

Likewise, $8 = 2^3$ three digit integers can be formed using only the digits 1 and 2. In the following circular arrangement, the first two digits of each each integer are the last two of its predecessor. For example, the last two digits of 111 are 11, and the first two digits of the subsequent number 112 are 11, and 121 follows 112, etc.

(b) Find a similar circular arrangement for the the $16 = 2^4$ four digit integers that can be formed with 1 and 2 so that the first three digits of each each integer are the last three of its predecessor.

	111	
211		112
221		121
222		212
	122	

PROBLEM 11: PERMUTATIONS FROM VERTICAL STACKS OF NUMBERS

A permutation of the integers $1, 2, 3, \ldots, n$ is simply a listing of these integers in some order. Thus, if $n = 5$, two permutations of 1, 2, 3, 4, 5 are: 4, 1, 2, 3, 5 and 3, 2, 1, 5, 4. It is well-known that the number of permutations on an n element set is $n! = n(n-1)(n-2)\cdots 2 \cdot 1$, i.e., the product of the first n integers.

The problem is to count the number of permutatations that satisfy an additional condition. Suppose we place the integers $1, 2, \ldots, n$ in several vertical stacks. For example, the integers from 1 to 5 can be placed in two stacks as on the left below.

1	
3	4
2	5

Next the first member of the permutation must be chosen from the top of one of the stacks. In this case, the first member of the permutation can be either 1 or 4. After a number has been chosen, it is removed from the stack, and the next number chosen must be from the top of one of the resulting stacks.

3	4
2	5

If for example, the first number chosen was 1, then the new stacks are those on the left. Thus, the next number chosen would have to be 3 or 4.

If the first number chosen had been 4, then the second choice would be either 1 or 5. Continuing, with the stacks as given above, 10 permutations could have been chosen: 1,3,2,4,5; 1,3,4,2,5; 1,3,4,5,2; 1,4,3,2,5; 1,4,3,5,2; 1,4,5,3,2; 4,5,1,3,2; 4,1,3,2,5; 4,1,3,5,2; 4,1,5,3,2.

How many stacked permutations are there on $\{1, 2, 3, 4, 5, 6\}$ with the stacks:

(a)

1	3	5
2	4	6

(b)

1	4
2	5
3	6

PROBLEM 12: THE ASSOCIATIVE LAW

Suppose that $a * b$ is some algebraic operation, that is, whenever a, b belong to some set S, then $a * b$ is also in S. The operation is associative if $(a * b) * c = a * (b * c)$.

A well-known example of a non-associative operation is exponentiation. Assuming that a, b are positive integers, then $a * b = a^b$ is also a positive integer, but $(a * b) * c \neq a * (b * c)$, except for a few values of a, b, c. For example, $(2 * 2) * 3 = (2^2)^3 = 2^6$ and $2 * (2 * 3) = 2^{(2^3)} = 2^8$.

If the associative law does not hold, then how a product $a_1 * a_2 * a_3 \cdots a_n$ is parenthesized is critical.

For example, with a product $a_1 * a_2 * a_3 * a_4$ of four terms there are five different parenthesizations. If the operation is exponentiation and, for example, $a_1 = 2$, $a_2 = 3$, $a_3 = 2$, $a_4 = 3$, each parenthesization gives a different answer:

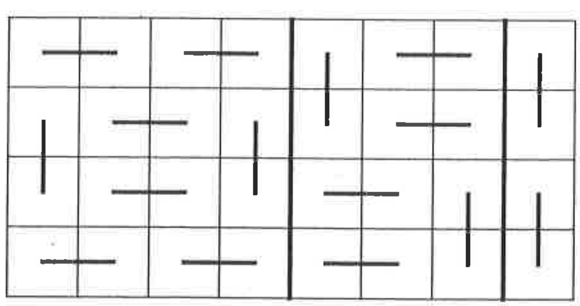
$((a_1 * a_2) * a_3) * a_4$ is $((2 * 3) * 2) * 3 = ((2^3)^2)^3 = (2^6)^3 = 2^{18}$;
$(a_1 * a_2) * (a_3 * a_4)$ is $(2 * 3) * (2 * 3) = (2^3)^{(2^3)} = (2^3)^8 = 2^{24}$;
$(a_1 * (a_2 * a_3)) * a_4$ is $(2 * (3 * 2)) * 3 = (2^9)^3 = 2^{27}$;
$a_1 * ((a_2 * a_3) * a_4)$ is $2 * ((3 * 2) * 3) = 2^{(9^3)} = 2^{(3^6)} = 2^{729}$;
$a_1 * (a_2 * (a_3 * a_4))$ is $2 * (3 * (2 * 3)) = 2^{(3^8)} = 2^{6561}$.

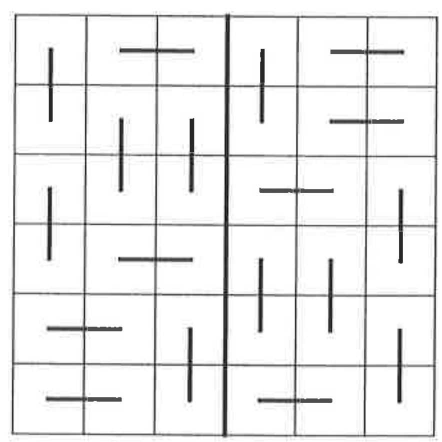
(a) How many ways are there to parenthesize an expression with five, six, or seven terms.
(b) Give an example, as the one above, to show that in the case of the five term expression, different parenthesizations give different answers.

PROBLEM 13: FAULT LINES IN TILINGS OF $M \times N$ GRIDS

The problem concerns covering an $m \times n$ grid with 1×2 tiles (dominoes) when m or n is an even integer. Below are given two examples. In the first a 4×8 grid is tiled with 1×2 tiles indicated by the dark lines between the centers of the grid squares. In the second a 6×6 grid is tiled.

Note that in both cases there is a fault line, indicated by a darkened grid line, that is not crossed by any of the 1×2 tiles. In fact, in the first case, there are two fault lines.





In all cases, prove your answer.
(a) Does a tiling of a $4 \times n$ grid always have a fault line?
(b) Does a tiling of a 6×6 grid always have a fault line?
(c) Does a tiling of a 6×8 grid always have a fault line?

PROBLEM 14: PRIME SUMS

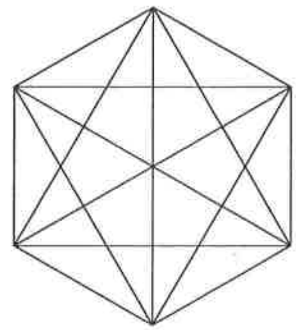
At the right the numbers $1, 2, 3, \ldots, 16$ have been placed in the squares of a 4×4 grid so that every pair of neighbors (in squares with a common edge) have sum that is a prime number. For example, going across the first row $13 + 16 = 29$, $16 + 15 = 31$, $15 + 14 = 29$, or down the first column, $13 + 6 = 19$, $6 + 5 = 11$, $5 + 8 = 13$, etc.

13	16	15	14
6	7	4	9
5	12	1	10
8	11	2	3

(a) Show, however, that the numbers $1, 2, 3, \ldots, 9$ cannot be placed on a 3×3 grid so that every pair of neighbors has sum that is prime.
(b) On the other hand, show that the numbers $1, 2, 3, \ldots, 12$ can be placed on a 3×4 grid so that every pair of neighbors has prime sum.
(c) Show that the numbers $1, 2, 3, \ldots, 24$ can be placed on a 4×6 grid so that every pair of neighbors has prime sum.

PROBLEM 15: THE NUMBER OF POINTS, LINES, AND TRIANGLES

To the right is a regular hexagon in which all possible lines have been drawn between the six vertices. There are 19 points (where two or more lines meet), 15 lines, and a careful count will give 98 triangles.



Suppose that all possible lines have been drawn between the eight vertices of a regular octagon. How many points, lines, and triangles are there?