Open Problem: Minimum Intersection of a Spanning Tree and a Perfect Matching

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Let $G = (V, E)$ be an unweighted, bipartite, cubic, 3-edge connected graph. We are interested in finding a TSP tour (i.e. a spanning, Eulerian subgraph) in G. By Petersen's theorem, G contains a perfect matching. Furthermore, if we remove a perfect matching from G, the remaining edges form a cycle cover. If the cycle cover has k components, then there is a TSP tour of length at most $n + 2k$: the cycle cover plus a double spanning tree on its k components form a spanning, Eulerian subgraph.

How can we determine the number of components in a cycle cover? One method is to use the size of the intersection of a spanning tree and a perfect matching to obtain an upper bound on the number of components in a cycle cover. If G contains a spanning tree T and a perfect matching M that intersect in k edges, then the number of cycles in $E \setminus M$ is at most k. In other words, the cycle cover obtained when M is removed from G has at most $|T \cap M|$ components. (In the more general case of bridgeless, cubic graphs, Mömke and Svensson implicitly showed that there exists a spanning tree T and a perfect matching M such that $|T \cap M| \leq n/6$ [MS11].)

Our goal is therefore to find trees and matchings with small intersections. Let us consider the following linear program for the TSP.

$$
\begin{array}{rrcll}
 & \min \quad \sum_{ij \in E} x_{ij} & & & \\
\forall S \subset V: & \sum_{i \in S, j \notin S} x_{ij} & \geq & 2, & \\
\forall ij \in E: & x_{ij} & \geq & 0. & \qquad \text{(P)}
\end{array}
$$

Recall the linear program for maximum matching in a bipartite graph.

$$
\begin{array}{rrcll}
 & \max \quad \sum_{ij \in E} y_{ij} & & & \\
\forall i \in V: & \sum_{j \in V} y_{ij} & = & 1, & \\
\forall ij \in E: & y_{ij} & \geq & 0. & \qquad \text{(Q)}
\end{array}
$$

Let $x^* \in \mathbb{R}^{|E|}$ be an optimal solution to (P) on G, and let $y^*_{ij} = 1 - x^*_{ij}$ for all $ij \in E$. Then we have the following straightforward claims, which make use of G being cubic, bipartite and 3-edge connected.

Lemma 1. *$x^*(E) = n$.*

Proof. Since G is 3-edge connected, each cut consists of at least three edges. Therefore, the solution $x_{ij} = 2/3$ for each $(i, j) \in E$ satisfies each cut and is a feasible solution with objective value n. □

Lemma 2. *y^* is a solution to (Q) on G with objective value $y^*(E) = \frac{n}{2}$.*

Proof. Since an optimal solution x^* has objective value n, for each vertex $v \in V$, $x^*(\delta(v)) = 2$. Furthermore, since G is a cubic graph, $y^*(\delta(v)) = 1$. Thus, $y^*(E) = n/2$. □

The point x^* can be decomposed into a convex combination of 1-trees and the point y^* can be decomposed into a convex combination of perfect matchings. If we were to find the 1-tree and the perfect matching from these decompositions with the smallest intersection, how small will it be? Fractionally, the points x^* and y^* have an intersection of zero, since for each edge $(i, j) \in E$, we have $x^*_{ij} + y^*_{ij} = 1$. Does this imply that there exists a spanning tree and perfect matching in G with a small intersection? In other words, is there an algorithm that rounds the LP relaxations (P) and (Q) to obtain a spanning tree and a perfect matching, respectively, while approximately preserving the small intersection?

References

[MS11] Tobias Mömke and Ola Svensson. Approximating graphic TSP by matchings. In *Foundations of Computer Science (FOCS), 2011 IEEE 52nd Annual Symposium on*, pages 560–569. IEEE, 2011.