6.966 Algorithmic Algebra with Applications Feb 19, 2008

Lecture 8

Lecturer: Neeraj Kayal *Scribe: Raga Velagapudi*

1 Linear Algebra Review Part 1

VECTORS. Let us define $\mathbb{F}^n \overset{\text{def}}{=} \{(a_1, ..., a_n) | a_i \in \mathbb{F}\}$. Then $\mathbb{F}^n$ is the set of n-dimensional vectors over the field $\mathbb{F}$. Given the above, a set of vectors $\mathbf{v}_1, ..., \mathbf{v}_n \in \mathbb{F}^n$ are said to be linearly dependent iff $\exists c_1, ..., c_n \in \mathbb{F}$ such that $\sum c_i \cdot v_i = 0$.

Another key concept is that of linear transformations. A linear transformation sends a point P in a vector space to a corresponding point Q in another vector space. For example, we can define linear transformation, ϕ, from $\mathbb{R}^2 \mapsto \mathbb{R}^2$ that takes an arbitary point (p, q) to $(p + q, p - q)$. Note that $\phi(0) = 0$ and $\phi(\alpha \cdot \mathrm{P} + \beta \cdot \mathrm{Q}) = \alpha\phi(\mathrm{P}) + \beta\phi(\mathrm{Q})$.

MATRICES. Two matrices M and M' are said to be similar, denoted $M \sim M'$, if

$$\exists \text{an invertible matrix } T, \text{ such that } T \cdot M \cdot T^{-1} = M'.$$

Note that this is equivalent to obtaining M' through a change in basis of the linear transformation corresponding to M.

The following is a key result in linear algebra:

Over any algebraically closed field $\mathbb{F}$ (for example $\mathbb{C}$), *almost every* square matrix M, is similar to a diagonal matrix D of the followig form:

$$\begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

Exercise 1. *Show that if*

$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix} \sim \begin{pmatrix} \beta_1 & 0 & 0 & 0 \\ 0 & \beta_2 & 0 & 0 \\ 0 & 0 & \beta_3 & 0 \\ 0 & 0 & 0 & \beta_4 \end{pmatrix}$$

then $\{\lambda_i | i \in [4]\} = \{\beta_i | i \in [4]\}$.

The following is the more formal description of the theorem:

Theorem 1. *Over an algebraically closed field $\mathbb{F}$ (for example, $\mathbb{C}$), any matrix $M \in \mathbb{F}^{n \times n}$ is similar to a block diagonal matrix D of the following form:*

$$\begin{pmatrix} B1 & 0 & \cdots & 0 \\ 0 & B2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_k \end{pmatrix}$$

where 0 *represents a zero matrix of the appropriate size and each* B_i *is a square block matrix of the following form*

$$B_i = \lambda_i \cdot I + T_i$$

where $\lambda_i \in \mathbb{F}$ *is some constant,* I *is the identity matrix and the* T_i*'s are matrices of the following form:*

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Note that every such T_i *is an upper triangular matrix whose entries are mostly zeroes. In fact all the entries on the diagonal are also zero. Only the entries on the* super-diagonal *are populated with* $1s$.

Exercise 2. *Show that if* T *is an* $m \times m$ *matrix of the above form then* $T^m = 0$.