Convergence in competitive games

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This talk is based on joint works with A. Vetta and with A. Sidiropoulos, A. Vetta
Cut game:

- Players: Nodes of the graph.
- Player’s strategy $\in \{1, -1\}$ (Republican or Democrat)
- An action profile corresponds to a cut.
- Payoff: Total Contribution in the cut.
- Change Party if you gain.

Cut Value: 7
2 and 5 are unhappy.
The Cut Game: Price of Anarchy

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Cut Value: 8
Pure Nash Equilibrium.
The Cut Game: Price of Anarchy

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2 and 5 are unhappy.

Cut Value: 12
The Optimum.

Social Function:
The cut value.

Price of Anarchy for this instance: \( \frac{12}{8} = 1.5 \).
Outline

- Performance in lack of Coordination: *Price of Anarchy*.
- **A Potential Game**: Cut Game
  - Lower Bounds: *Long poor paths*
  - Upper Bounds: *random paths*
- **Basic-utility and Valid-utility Games**
  - Basic-utility Games: Fast Convergence.
  - Valid-utility Games: *Poor Sink Equilibria*
- Conclusion: Other Games?
We can model selfish behavior of players by a sequence of best responses by players.
Convergence to Approximate Solutions

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How fast do players converge to an approximate solution?

Our goal: How fast do players converge to an approximate solution?
In a **fair path**, we should let each player play at least once after each polynomially many steps.
Fair Paths

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- Random path: We pick the next player at random.
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We are interested in the Social Value at the end of a fair path.
A Cut game: The Party Affiliation Game

Cut game:

Social Function:
- The Cut Value
- Total Happiness

Price of anarchy: at most 2.

Local search algorithm for Max-Cut!
A Cut game: The Party Affiliation Game

Cut game:

Social Function:
- The Cut Value

Convergence:
- Finding local optimum for Max-Cut is PLS-complete (Schaffer, Yannakakis [1991]).

Cut Value: 7
2 and 5 are unhappy.
Cut Game: Paths to Nash equilibria

- **Unweighted graphs**: After $O(n^2)$ steps, we converge to a Nash equilibrium.

- **Weighted graphs**: It is PLS-complete.
  - PLS-Complete problems and tight PLS reduction (Johnson, Papadimitriou, Yannakakis [1988]).
  - Tight PLS reduction from Max-Cut (Schaffer, Yannakakis [1991])
  - There are some states that are exponentially far from any Nash equilibrium.

**Question**: Are there long poor fair paths?
Cut Game: A Bad Example

Consider graph $G$, a line of $n$ vertices. The weight of edges are $1, 1 + \frac{1}{n}, 1 + \frac{2}{n}, \ldots, 1 + \frac{n-1}{n}$. Vertices are labelled $1, \ldots, n$ throughout the line. Consider the round of best responses:
A Bad Example: Illustration

After one move.
A Bad Example: Illustration

After two moves.
A Bad Example: Illustration

After $n$ moves (one round)
A Bad Example: Illustration

After two rounds.

Theorem: In the above example, the cut value after $k$ rounds is $O\left(\frac{k}{n}\right)$ of the optimum.
Random One-round paths

Theorem: (M., Sidiropoulos[2004]) The expected value of the cut after a random one-round path is at most $\frac{1}{8}$ of the optimum.
Random One-round paths

**Theorem:** (M., Sidiropoulos[2004]) The expected value of the cut after a random one-round path is at most \( \frac{1}{8} \) of the optimum.

**Proof Sketch:** The sum of payoffs of nodes after their moves is \( \frac{1}{2} \)-approximation. In a random ordering, with a constant probability a node occurs after \( \frac{3}{4} \) of its neighbors. The expected contribution of a node in the cut is a constant-factor of its total weight.
Theorem: (M., Sidiropoulos[2004]) There exists a weighted graph $G = (V(G), E(G))$, with $|V(G)| = \Theta(n)$, and exponentially long fair path such that the value of the cut at the end of $P$, is at most $O(1/n)$ of the optimum cut.
Exponentially Long Poor Paths

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**Proof Sketch:**
Use the example for the exponentially long paths to the Nash equilibrium in the cut game. Find a player, $v$, that moves exponentially many times. Add a line of $n$ vertices to this graph and connect all the vertices to player $v$. 
Poor Long Path: Illustration
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Mildly Greedy Players

A Player is 2-greedy, if she does not move if she cannot double her payoff.
Mildly Greedy Players

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- **Theorem:** (M., Sidiropoulos[2004]) One round of selfish behavior of 2-greedy players converges to a constant-factor cut.

- Proof Idea: If a player moves it improves the value of the cut by a constant factor of its contribution in the cut.
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A Cut game: Total Happiness

Cut game:
- The happiness of player $v$ is equal to his total contribution in the cut minus the weight of its adjacent edges not in the cut.

Social Function:
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Social Function:

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- In the context of correlation clustering: Maximizing agreement minus disagreement (Bansal, Blum, Chawla[2002]).

$\log n$-approximation algorithm is known. (Charikar, Wirth[2004]).
A Cut game: Total Happiness

- Cut game:
  - The **happiness** of player $\nu$ is equal to his total contribution in the cut minus the weight of its adjacent edges not in the cut.

- Social Function:
  - Total Happiness: Sum of happiness of players
  - **Price of anarchy**: unbounded in the worst case.
  - A bad example: a cycle of size four.
A Cut game: Total Happiness

Cut game:

- The happiness of player $v$ is equal to his total contribution in the cut minus the weight of its adjacent edges not in the cut.

Social Function:

- Total Happiness: Sum of happiness of players
- The expected happiness of a random cut is zero.

Our result: For unweighted graphs of large girth, if we start from a random cut, then after a random one-round path, the expected happiness is a sublogarithmic-approximation.
Cut Game: Total Happiness

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For a pair $u, v \in V(G)$, let $E_{u,v}$ denote the event that there exists a path $p = x_1, x_2, \ldots, x_{|p|}$, with $u = x_1$, and $v = x_{|p|}$, and for any $i$, with $1 \leq i < |p|$, $x_i < x_{i+1}$. 
For some $\delta > 0$, we call an edge of $G$, $\delta$-good, if at least one of its end-points, has degree at most $\delta$.

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**Lemma:** Let $\{u, v\}, \{v, w\} \in E(G)$, such that $u \prec w \prec v$. There exists a constant $C$, such that if the girth of $G$ is at least $C \frac{\log n}{\log \log n}$, then $\Pr[E_{u,w}] < n^{-3}$.
Cut Game: Total Happiness

For some $\delta > 0$, we call an edge of $G$, $\delta$-good, if at least one of its end-points, has degree at most $\delta$.

For a pair $u, v \in V(G)$, let $\mathcal{E}_{u,v}$ denote the event that there exists a path $p = x_1, x_2, \ldots, x_{|p|}$, with $u = x_1$, and $v = x_{|p|}$, and for any $i$, with $1 \leq i < |p|$, $x_i < x_{i+1}$.

Lemma: Let $\{u, v\}, \{v, w\} \in E(G)$, such that $u < w < v$. There exists a constant $C$, such that if the girth of $G$ is at least $C \frac{\log n}{\log \log n}$, then $\Pr[\mathcal{E}_{u,w}] < n^{-3}$.

Lemma: For any $e \in E(G')$, we have $\Pr[e$ is cut $] \geq 1/2 - o(1)$. 
Lemma: Let \( e = \{u, v\} \in E(G) \), with \( u < v \), and \( \deg(v) \leq \delta \). Then, \( \Pr[e \text{ is cut}] \geq 1/2 + \Omega(1/\sqrt{\delta}) \).
Lemma: Let $e = \{u, v\} \in E(G)$, with $u < v$, and $\deg(v) \leq \delta$. Then, $\Pr[e$ is cut $] \geq 1/2 + \Omega(1/\sqrt{\delta})$.

Theorem: (M., Sidiropoulos[2004]) There exists a constant $C'$, such that for any $C > C'$, and for any unweighted simple graph of girth at least $C \frac{\log n}{\log \log n}$, if we start from a random cut, the expected value of the happiness at the end of a random one-round path, is within a $\frac{1}{(\log n)^{O(1/C)}}$ factor from the maximum happiness.
Outline

- Performance in lack of Coordination: Price of Anarchy.
- State Graphs, Convergence, and Fair Paths.
- Cut Games: Party Affiliation Games
  - Lower Bounds: Long poor paths
  - Upper Bounds: random paths
  - Total Happiness: Cut minus Other Edges
- Basic-utility and Valid-utility Games.
  - Basic-utility Games: Fast Convergence.
  - Valid-utility Games: Poor Sink Equilibria!
- Conclusion: Other Games?
Valid-Utility Games

- Ground Set of Markets: $V = \{v_1, v_2, \ldots, v_n\}$.
- Player $i$ can provide a subset of $V$. $S_i$ is a family of subsets of $V$ feasible for player $i$.
- $S_i \subset V$ is the strategy of player $i$. $S_i \in S_i$. 
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Social Function:

A submodular set function \( f: 2^V \rightarrow R \) on union of strategies: \( f(\bigcup_{1 \leq i \leq n} S_i) \).
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- Social Function:
  - A submodular set function \( f : 2^V \rightarrow R \) on union of strategies: \( f(\bigcup_{1 \leq i \leq n} S_i) \).
- The payoff of any player is at least the change that he makes in the social function by playing.
- The sum of payoffs is at most the social function.
- Several examples, including the market sharing game and a facility location game.
Valid-Utility Games

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The sum of payoffs is at most the social function.

In basic-utility games, the payoff is equal to the change that a player makes.
Example: Market Sharing Game

- Market Sharing Game
  - $n$ markets and $m$ players.
  - Market $i$ has a value $q_i$ and cost $C_i$.
  - Player $j$ has a budget $B_j$.
  - Player $j$’s action is to choose a subset of markets of his interest whose total cost is at most $B_j$.
  - The value of a market is divided equally between players that provide these markets.
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- The value of a market is divided equally between players that provide these markets.

Social Function: Total query that’s satisfied in the market. (submodular.)
Valid-utility Games: Price of Anarchy

- **Theorem:** (Vetta[2002]) The price of anarchy (of a mixed Nash equilibrium) in valid-utility games is at most 2.

- **Theorem:** (Vetta[2002]) Basic-utility games are potential games. In particular, best responses will converge to a pure Nash equilibrium.

- **Theorem:** (Goemans, Li, Mirrokni, Thottan[2004]) Pure Nash equilibria exist for market sharing games and can be found in polynomial time in the uniform case.
Basic-Utility Games : Convergence

Theorem: (M., Vetta[2004]) In basic-utility games, after one round of selfish behavior of players, they converge to a $\frac{1}{3}$-optimal solution.
**Market Sharing Games : Convergence**

- **Theorem:** (M., Vetta[2004]) In basic-utility games, after one round of selfish behavior of players, they converge to a $\frac{1}{3}$-optimal solution.

- **Theorem:** (M., Vetta[2004]) In a market sharing game, after one round of selfish behavior of players, they converge to a $\frac{1}{\log(n)}$-optimal solution and this is almost tight.
Valid-utility Games: Convergence

- **Theorem:** (M., Vetta[2004]) For any $k > 0$, in valid-utility games, the social value after $k$ rounds might be $\frac{1}{n}$ of the optimal social value.
Sink Equilibria

A sink equilibrium is a minimal set of states such that no best response move of any player goes out of these states.
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If we enter a sink equilibrium, we are stuck there. Even random best-response paths cannot help us going out of a sink equilibrium.

Price of anarchy for sink equilibria vs. the price of anarchy for Nash equilibria.
Sink Equilibria

Theorem: (M., Vetta) In valid-utility games, even though the price of anarchy for Nash equilibria is $\frac{1}{2}$, the price of anarchy for sink equilibria is $\frac{1}{n}$.

The performance of the Nash equilibria (or the price of anarchy for NE) is not a good measure for these games.

Theorem: (M., Vetta) Finding a sink equilibrium in valid-utility games is PLS-Hard and there are states that are exponentially far from any sink equilibria.
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Conclusion

- Study **Speed of convergence to approximates solutions** instead of to Nash equilibria.
- **Sink equilibria:** an alternative measure to study the performance of the systems in lack of coordination.
Open problems

- Are there exponentially long fair paths in Basic-utility games.

- Is finding a 2-approximate Nash equilibrium for the cut game in P? How long does it take that 2-greedy players converge to a (2-approximate) Nash equilibrium? If it is polynomial, then finding a 2-approximate Nash equilibrium is in P.

- Are there exponentially long paths in the market sharing game?

- Study covering and random paths in other games.