CONVERGENCE OF GRAPHS

AND

GENERALIZED QUASIRANDOM GRAPHS

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When are two (large) graphs similar

- close to each other
- local $\leftrightarrow$ global properties
- approximation of large $g$ by small $g$. 
Generalized quasirandom graph
Convergence of graph sequences
Limit objects
Distance, completion
Approximation; Szemerédi–lemma
\((G_n) \text{ converges} \sim \)

\[ A \text{ for "density" of } F \text{ in } (G_n) \]

converges for \( n \to \infty \)
\((G(n;p))\) sequence of \(p\) - random graphs

\[ m(F, G) = \# \text{ labeled copies of } F \text{ in } G \]
\[ m(F, G_n) \sim n^k p^e \]

Def. \((G_n)\) is a sequence of \(p\) - quasirandom graphs if \(\forall F\)

\[ \frac{m(F, G_n)}{n^k} \to p^e \]

\((\star)\)

Theorem (Chung, Graham, Wilson)

\[ \frac{m(\rightarrow, G_n)}{n^2} \to p \]
\[ \frac{m(\square, G_n)}{n^4} \to p^4 \]

\(\Rightarrow (G_n)\) is \(p\) - quasirandom

\((\star)\) holds \(\forall F\)
Generalized random graphs

A weighted graph

\[ V(H) = \{1, \ldots, q\}, \quad \alpha_i > 0, \]

vertex weights \( \alpha_i > 0 \),

edges weights \( 0 \leq \beta_{ij} \leq 1 \)

We may assume: \( H \) complete, with loop

Generalized random graph with model \( H \)

\[ \sum_i \alpha_i = 1 \]

\[ |V_i| = \alpha_i n \]

\[ 1 \leq i \leq q \]

\( G(n; H) \)
Def \( (G_n) \)

generalized \( H \)-quasirandom

\[ \forall F \]

\[ m(F, G_n) \sim m(F, G(n; H)) \] \(*\)

# copies of \( F \subset G_n \)

**Questions**

Is the structure of \( G_n \) similar to \( G(n; H) \)?

Is it enough to require \((*)\) ?

For a finite set \( \{F_i\} \)

(depending on \( H \))
If all the vertex-weights and edge-weights are 1, (no edge: 0):
\[ \text{hom}(F, H) = \# \varphi : V_F \rightarrow V_H \]
\[ \downarrow \]
\[ \text{homomorphism, edge-preserving} \]

\[ t(F, H) = \frac{\text{hom}(F, H)}{(\sum \alpha_i)^{V_F}} \]

\[ F \text{ simple, } G \text{ simple, } \text{unweighted} \]

\[ t(F, G) = \frac{\text{hom}(F, G)}{n^R} \]

\[ n = |V_G| \]
\[ R = |V_F| \]

\[ \text{density of } F \text{ in } G \]
Def \( H \) "small", weighted

\((G_n) \) is \( H \)-quasirandom, if

\[
\forall F \quad t(F, G_n) \rightarrow t(F, H)
\]

Exp \( G(n; H) \) \( H \)-random sequence

\[
t(F, G(n; H)) \rightarrow t(F, H)
\]

with prob. 1
Theorem (Lovász-S.)

\((G_n)\) \(H\)-quasirandom

structure of \(G_n \sim\) structure of \(G(n; H)\)

\((G_n)\) \(H\)-quasirandom. Then

\(\forall n \ \exists \ \mathcal{V}_{G_n} = \bigcup V_i\) such that

\[ \frac{|V_i|}{|V_{G_n}|} \rightarrow \alpha_i \quad 1 \leq i \leq q \]

\(G_n(V_i)\) is \(\beta_{ii}\)-quasirandom

\(G_n(V_i, V_j)\) is \(\beta_{ij}\)-quasirandom

bipartite

\(\exists\) finite test-class

\((G_m)\) is \(H\)-quasirandom iff

\[ t(F, G_n) \rightarrow t(F, H) \]

for \(\forall F\) with \(|V_F| < (10q)^q\)

where \(a = |V_H|\)
minimal finite family $F$.

structure,

$|F|$.
CONVERGENCE OF GRAPHS

\((G_n)\), \(F, G_n\) simple

\[ t(F, G_n) = \frac{\text{hom}(F, G_n)}{|V_{G_n}|/|V_F|} \]

**Def** \((G_n)\) is convergent, if

\[ \forall F \quad t(F, G_n) \quad \text{is convergent} \]

**Exp** \((G(n;p))\) \(p\)-random

\[ t(F, G_n) \rightarrow p \left| E_F \right| \quad \text{with prob.} \]

\((G(n;p))\) \(p\)-quasirandom

\[ t(F, G_n) \rightarrow p \left| E_F \right| \]

**Exp** \((G_n)\) generalized \(H\)-quasirandom

\[ -12- \quad t(F, G_n) \rightarrow t(F, H) \]
What is the limit?

$(G_n)$ p-random

$(G_n)$ p-quasirandom

$(G_n)$ H-random

$(G_n)$ H-quasirandom

$t(F,G_n) \rightarrow t(F,H)$

Is $H$ the limit?

$(G_n)$ convergent, but not $H$-quasir.
Lovász - B. Szegedy

Let

\[ \mathcal{W} = \{ w : [0,1]^2 \to [0,1], \text{ meas., symm.} \} \]

\[ t(F, w) = \int_{[0,1]^k} \prod_{i,j \in E_F} w(x_i, x_j) \, dx \]

\[ k = |E_F| \]
\( G_n \leftrightarrow \omega G_n \)

\[
t (F, G) = t (F, \omega G)
\]

\[
\omega = \begin{cases} 
1 & \text{if } (i,j) \in E_G \\
0 & \text{if } (i,j) \notin E_G
\end{cases}
\]
Theorem (Lovász - B. Szegedy)

- For every convergent graph sequence $(G_n)$, there is a $w \in W$ such that

\[ \forall F \quad \lim_{n \to \infty} t(F, G_n) = t(F, w) \]

- $\forall w \in W$ arises as a limit of some sequence $(G_n)$

Borgs - Chayes - Lovász

$W$ determined up to mean-pres. transform.

\[ \forall F \quad \lim_{n \to \infty} t(F, G_n) = t(F, w) \]

\[ \lim G_n = W \]

\[ \square \]
Let $H$ be a weighted graph
\[ \omega_H \in \mathcal{W} \]
with
\[ \sum \omega_i = 1 \]

Step functions

\[ G(n; p), \]
\[ (G_n)_{p\text{-quasirandom}} \]
\[ \xrightarrow{\text{S\'emeredi-lemma}} \]
\[ \lim = \omega_p = p \]
\[ \text{const. f.} \]

\[ G(n; H), \]
\[ (G_n)_{H\text{-random}} \]
\[ \xrightarrow{\text{approximation by step-function}} \]
\[ \lim = \omega_H \]
\[ \text{step f.} \]
DISTANCE OF GRAPHS \\
\text{\textsc{w - metric space}}

1. \quad V_G = V_{G'} \\
\quad d_\square (G, G') = \max_{s, T \in V} \frac{1}{|V_G|^2} \left| e_G(s, T) - e_{G'}(s, T) \right|

2. "best overlay" , \( |V_G| = |V_{G'}| \) \\
\quad \delta_\square (G, G') = \min_{\tilde{G} \cong G} \ d_\square (\tilde{G}, G')

3. "best fractional overlay" \\
also for \( |V_G| \neq |V_{G'}| \), \\
\quad \delta_\square (G, G') = \min_X \ d_\square (G(X), \tilde{G}'(X))

\text{where} \\
\quad \sum_{i=1}^{n} X_{iu} = \alpha_i (G) , \quad \sum_{i=1}^{n} X_{iu} = \alpha_u (G')
\((G_n)\) is convergent \(\iff\) \((G_n)\) is Cauchy in \(\mathcal{D}_\Box\) metric

\(G\) similar \(G'\)

\(\sim\) close in \(\mathcal{D}_\Box\)

Szemeredi - Lemma

\(\sim\) approximation by step function