MAY’S THEOREM FOR TREES

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1. Introduction

In 1952, Kenneth May gave an elegant characterization of simple majority decision based on a set with exactly two alternatives [9]. This work is a model of the classic voting situation where there is two candidates and the candidate with the most votes is declared the winner. May’s theorem is a fundamental result in the area of social choice and it has inspired many extensions. See [2], [3], [4], [5], [8], and [10] for a sample of these results.

The goal of the current paper is to state and prove a version of May’s theorem in the context of trees. In what follows, tree will mean a rooted tree with labelled leaves and unlabelled interior vertices, and no vertex except possibly the root can have degree 2. In the biological literature, such a tree $T$ might represent the evolutionary history of the set $S$ of species, with interior vertices of $T$ representing ancestors of the species in $S$. Clearly the simplest nontrivial case is when $|S| = 3$. In this case, there are exactly 4 distinct trees with leaves labelled by the set $S$.

It is within this context that we define a version of simple majority decision for trees and characterize it in terms of three conditions. There is a clear connection between our conditions and those given by May.

This paper is divided into four sections with this introduction being the first section. Section 2 is background material on May’s work and includes the statement of May’s Theorem. Section 3 contains the definition of majority decision for trees, and the main result of this paper is stated and proved in Section 4.

2. Background on May’s Work

Let $S = \{x, y\}$ be a set with two alternatives. The binary relations $R_{-1} = \{(x, x), (y, y), (y, x)\}$, $R_0 = S \times S$, and $R_1 = \{(x, x), (y, y), (x, y)\}$ are the three weak orders on $S$. The relation $R_{-1}$ represents the situation where $y$ is strictly preferred to $x$, $R_1$ represents the situation where $x$ is strictly preferred to $y$, and $R_0$ represents indifference between $x$ and $y$.

Let $K = \{1, \ldots, k\}$ be a set with $k \geq 2$ individuals and let $\mathcal{W}(S)$ be the set $\{R_{-1}, R_0, R_1\}$. A function of the form

$$f : \mathcal{W}(S)^k \rightarrow \mathcal{W}(S)$$

is called a group decision function by May.

For any $p = (D_1, \ldots, D_k)$ in $\mathcal{W}(S)^k$ and for any $i \in \{-1, 0, 1\}$ let

$$N_p(i) = |\{D_j : D_j = R_i\}|.$$ 

That is, $N_p(i)$ is the number of times the relation $R_i$ appears in the $k$-tuple $p$. It follows that $N_p(-1) + N_p(0) + N_p(1) = k$ and $N_p(i) \geq 0$ for each $i \in \{-1, 0, 1\}$. 

The group decision function
\[ M : \mathcal{W}(S)^k \to \mathcal{W}(S) \]
defined by
\[
M(p) = \begin{cases} 
R_{-1} & \text{if } N_p(1) - N_p(-1) < 0 \\
R_1 & \text{if } N_p(1) - N_p(-1) > 0 \\
R_0 & \text{if } N_p(1) - N_p(-1) = 0
\end{cases}
\]
for any \( k \)-tuple \( p \) is called, for obvious reasons, simple majority decision. The consensus weak order \( M(p) \) has \( y \) strictly preferred to \( x \) if more individuals rank \( y \) strictly over \( x \) than \( x \) strictly over \( y \). There is indifference between \( x \) and \( y \) if the number of individuals that strictly prefer \( y \) over \( x \) is the same as the number of individuals that strictly prefer \( x \) over \( y \). Finally, \( M(p) \) has \( x \) strictly preferred to \( y \) if the number of individuals that rank \( x \) strictly over \( y \) is more than the number of individuals that rank \( y \) strictly over \( x \).

May simplified the notation used above as follows. The relation \( R_{-1} \) is identified with the number \(-1\), the relation \( R_0 \) is identified with the number \(0\), and the relation \( R_1 \) is identified with \(1\). Using this identification we can think of a group decision function as a function with domain \( \{-1, 0, 1\}^k \) and range \( \{-1, 0, 1\} \).

Let \( f : \{-1, 0, 1\}^k \to \{-1, 0, 1\} \) be a group decision function. Then reasonable properties that \( f \) may or may not satisfy are the following.

(A) For any \( k \)-tuple \( p = (D_1, \ldots, D_k) \) and for any permutation \( \alpha \) of \( K \),
\[
f(D_{\alpha(1)}, \ldots, D_{\alpha(k)}) = f(D_1, \ldots, D_k).
\]

(N) For any \( k \)-tuple \( p = (D_1, \ldots, D_k) \),
\[
f(-D_1, \ldots, -D_k) = -f(D_1, \ldots, D_k).
\]

(PR) For any \( k \)-tuples \( p = (D_1, \ldots, D_k) \) and \( p' = (D'_1, \ldots, D'_k) \),

if \( f(D_1, \ldots, D_k) \in \{0, 1\}, D'_i = D_i \) for all \( i \neq i_0 \), and \( D'_{i_0} > D_{i_0} \),

then
\[
f(D'_1, \ldots, D'_k) = 1.
\]

The conditions (A), (N), and (PR) correspond to conditions II, III, and IV given on pages 681 and 682 in [9]. Condition (A) states that \( f \) is a symmetric function of its arguments and thus individual voters are anonymous. Condition (N) is called neutrality. This axiom is motivated by the idea that the consensus outcome should not depend upon any labelling of the alternatives. Condition (PR) is called positive responsiveness since it reflects the notion that a group decision function should respond in a positive way to changes in individual preferences. If the consensus outcome \( f(p) \) does not rank \( y \) strictly preferred to \( x \) and one individual \( i_0 \) changes their vote in a favorable way toward \( x \), then the consensus outcome \( f(p') \) should strictly prefer \( x \) to \( y \).

We now can state May’s result.

**Theorem 1.** A group decision function is the method of simple majority decision if and only if it satisfies (A), (N), and (PR).
3. Trees with 3 Leaves

As we have noted, May studied majority decision for two alternatives, which is the simplest non-trivial case for weak orders. Since our goal is to prove a version of May’s result for trees, we too restrict our attention to the simplest non-trivial case for trees; namely when \(|S| = 3\). For \(S = \{x, y, z\}\), and \(\{u, v\} \subset S\), let \(T_{\{u,v\}}\) denote the tree with one non-root vertex of degree three adjacent to the root, \(u\), and \(v\). Let \(T_\emptyset\) be the tree whose only internal vertex is the root.

Let \(T(S)\) be the set \(\{T_{\{x,y\}}, T_{\{x,z\}}, T_{\{y,z\}}, T_\emptyset\}\) of all trees with the leaves labelled by the elements of \(S\). We will call a function of the form \(C : T(S)^k \rightarrow T(S)\) a consensus function to conform with current usage \([6]\). An element \(P = (T_1, \ldots, T_k)\) in \(T(S)^k\) is called a profile and the output \(C(P)\) is called a consensus tree.

For any profile \(P = (T_1, \ldots, T_k)\) and for any two element subset \(\{u, v\}\) of \(S\), let

\[ N_P(uv) = |\{T_i : T_i = T_{\{u,v\}}\}|. \]

Also, let

\[ N_P(\emptyset) = |\{T_i : T_i = T_\emptyset\}|. \]

So \(N_P(xy) + N_P(xz) + N_P(yz) + N_P(\emptyset) = k\). The consensus function

\[ Maj : T(S)^k \rightarrow T(S) \]

defined by

\[ Maj(P) = \begin{cases} T_{\{u,v\}} & \text{if } N_P(uv) > \frac{k}{2} \\ T_\emptyset & \text{otherwise} \end{cases} \]

is called majority rule \([7]\). This consensus function is well known but it is not the best analog of simple majority decision sense May. We feel that a better candidate is the consensus function

\[ M : T(S)^k \rightarrow T(S) \]

defined by

\[ M(P) = \begin{cases} T_{\{u,v\}} & \text{if } N_P(uv) > \max\{N_P(uw), N_P(vw)\} \\ T_\emptyset & \text{otherwise} \end{cases} \]

where \(\{u, v, w\} = \{x, y, z\}\). It is easy to see that if \(Maj(P) = T_{\{u,v\}}\) for some two element subset \(\{u, v\}\) of \(S\), then \(M(P) = Maj(P)\). The converse is not true. For example, if \(P = (T_1, \ldots, T_k)\) such that \(T_1 = T_{\{x,y\}}\) and \(T_i = T_\emptyset\) for all \(i \neq 1\) in \(K\), then \(M(P) = T_{\{x,y\}}\) and \(Maj(P) = T_\emptyset\). For the remainder of this paper the function \(M\) will be called majority decision.

4. Main Result Summary

We introduce translations of the conditions \((A), (N), \text{ and (PR)} [^+]\) to the context of trees and prove

**Theorem 2.** The consensus function \(C : T(S)^k \rightarrow T(S)\) is simple majority decision if and only if \(C\) satisfies \((A)^+, (N)^+, \text{ and (PR)}^+\).
REFERENCES


