On the Capacity of Information Networks

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“There is as yet no unified theory of network information flow. But there can be no doubt that a complete theory of communication networks would have wide implications for the theory of communication and computation.”
- Cover & Thomas, *Elements of Information Theory*. 
History of Network Coding

- Breakthrough [Ahlsweede et al. ’00].
  - Existence of multicast solution depends on min-cut condition.
- Algebraic framework [Koetter & Médard ’03].
  - Led to a randomized, distributed, fault-tolerant algorithm for multicast [Ho et al. ’03].
- Deterministic algorithms for multicast [Jaggi et al. ’03, Harvey et al. ’05].
The Network Coding Problem

Given:

- Directed acyclic graph $G$.
- Integral capacity $c(u, v)$ for each edge $(u, v)$.
- $k$-commodities:
  - Set of source nodes.
  - Set of sink nodes.
The Idea of Network Coding

- There is one message for each commodity.
  - Every source knows the message.
  - Every sink wants the message.
  - A message is a single symbol from an alphabet $\Sigma$.
- Each edge of capacity $c$ can transmit $c$ symbols from $\Sigma$.
- Question: Does there exist a solution?
This Talk: from Existence to Optimization

- Consider size of alphabet $\Sigma$.
  - Model of network coding that works for multicast doesn’t work well in general.
  - Need a notion of “rate”.
- What is the maximum achievable communication rate in a network?
  - Explore bounds based on cut conditions.
  - Develop entropy inequalities based on graph structure.
- What is the maximum rate in an undirected network?
Alphabet Size
Who Cares About Alphabet Size?

• Small alphabet means simple, efficiently-computable edge functions.
• Large alphabet implies large latency.
• Need $\Omega(\log |\Sigma|)$ bits of memory at each node to compute edge functions (naively).
• An upper bound on $|\Sigma|$ would imply that the network coding problem is decidable.
Our Results - The Bad News

• Sometimes an enormous alphabet is required!
  ◦ An $n$-node network may require an alphabet of size:
    \[
    |\Sigma| = 2^{e^{\Omega(n^{1/3})}}
    \]
  ◦ Solution may exist but be hopelessly unwieldy!

• Nonmonotonicity:
  ◦ Instance solvable with 4-symbol alphabet, but not with 1000-symbol alphabet!
  ◦ Can’t fix a single large alphabet size, e.g. $2^{64}$. 
Lemma 1  \textbf{Solvable iff} $|\Sigma| = q^k$. 

\textbf{Building Block: Network } $I_k$

has messages $M_1, \ldots, M_k, P_1, \ldots, P_k$

\begin{itemize}
\item \text{wants all } M\text{'s & } P\text{'s}
\item \text{wants all } M\text{'s}
\item \text{wants all } M\text{'s}
\item \text{wants all } M\text{'s}
\item \text{-}M_i + P_j
\item \text{-}P_j + M_i
\end{itemize}

capacity $2k-2$
capacity $k-1$
capacity $2$
Doubly-Exponential Lower Bound

- Network $I_k$ has $O(k^2)$ nodes and requires $|\Sigma|$ to be a perfect $k$-th power.

- Let $J_n$ consist of disjoint networks

  $$I_2 \ I_3 \ I_5 \ I_7 \ I_{11} \ \ldots \ \ I_p$$

  where $p$ is largest prime less than $n^{1/3}$.

$\Rightarrow$ $J_n$ has $O(n)$ nodes and there is a solution if and only if:

$$|\Sigma| = C^2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot \ldots \cdot p \ = \ C e^{\Omega(n^{1/3})}$$

$$\geq 2 e^{\Omega(n^{1/3})}$$
Our Results - The Good News

If each edge can send one additional bit, then the minimum alphabet size is $O(1)$.

- Our bad example is an artifact of using the network at 100.0% capacity.
- Are we wasting our time with this model?
- **Tweak the model?**
  - Messages are drawn from an alphabet $\Gamma$.
  - Each edge transmits one symbol from larger alphabet $\Sigma$.
  - Rate $= \frac{\log |\Gamma|}{\log |\Sigma|}$. 
What is the Maximum Achievable Rate?
What is the Maximum Achievable Rate?

- Open problem except for multicast where max rate = min-cut between the source and any sink.
- Is there a cut-based upper bound on rate for the general problem?
- Do information theoretic tools give a better upper bound?
Sparsity

- Sparsity of a cut $A \subseteq E$ is:

\[
\text{capacity of edges in cut } A \\
\text{\# commodities with no remaining source-sink path}
\]

- Sparsity of a graph is minimum sparsity over all cuts.
- There exist directed graphs in which the maximum rate $> \text{sparsity}$.

![Diagram of a directed graph with sparsity highlighted]

Sparsity = $1/2$
Rate = 1
Meagerness

- A set of commodities $P$ is *separated* by a cut if there is no remaining path from a source of *any* commodity in $P$ to a sink of *any* commodity in $P$.

- The *meagerness* of a graph is the minimum over all sets of commodities $P$ and cuts that separate $P$ of
  \[
  \frac{\text{capacity of edges in cut}}{|P|}
  \]

- The maximum rate $\leq$ meagerness in directed graphs.

\[
\text{Meagerness} = 1 \\
\text{Rate} = 1
\]
Sometimes Max Rate $< \text{Meagerness}$
Sometimes Max Rate < Meagerness

- The meagerness is 1.
Sometimes Max Rate $< \text{Meagerness}$

- The meagerness is 1.
- This flow solution has rate $2/3$. Best possible?

\[
\Gamma = \{0,1\}^2
\]
\[
\Sigma = \{0,1\}^3
\]
Better Bounds Through Entropy

* Obtain strictly better bounds on rate through *entropy* arguments.
  - Show max rate $2/3$ for previous example.
  - Implies meagerness is a loose upper bound on rate.

* Entropy of a random variable $X$ is the information in $X$ measured in bits.
  - The entropy of $X$ is denoted $H(X)$.
  - The entropy of $X$ and $Y$ together is $H(X, Y)$. 
Entropy View of Network Coding

• Suppose messages are selected independently and uniformly from $\Gamma$.
• As a result, the symbol transmitted on each edge is a R.V.
• Structure of graph and properties of entropy imply constraints that a network code must satisfy.
Entropy and Network Coding

- Properties of entropy:
  - Nonnegative: \( H(U) \geq 0 \).
  - Nondecreasing: \( H(U, x) \geq H(U) \).
  - Submodular: \( H(U) + H(V) \geq H(U \cup V) + H(U \cap V) \).

- Constraints on a network coding solution:
  - Uniformity of sources: \( H(S_A) = \log |\Gamma| \).
  - Independence of sources: \( H(S_A, S_B) = H(S_A) + H(S_B) \).
  - Sources = sinks: \( H(S_A, U) = H(T_A, U) \) for all \( U \).
  - Edge capacity: \( H(e) \leq \log |\Sigma| \).
One More Condition: Downstreamness

$U$ is downstream of $V$ if all paths from a source to an edge in $U$ intersect $V$.

If $U$ is downstream of $V$, $H(V) = H(U, V)$.

Ex 1: $T_b$ is downstream of $\{S_a, F\}$. $H(S_a, F) = H(S_a, T_b, F)$.
One More Condition: Downstreamness

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If $U$ is downstream of $V$, 
\[ H(V) = H(U, V). \]

Ex 1: $T_b$ is downstream of $\{S_a, F\}$. 
\[ H(S_a, F) = H(S_a, T_b, F). \]

Ex 2: $T_a$ is downstream of $\{S_b, G\}$. 
\[ H(S_b, G) = H(T_a, S_b, G). \]
One More Condition: Downstreamness

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$H(S_a, F) = H(S_a, T_b, F)$.

Ex 2: $T_a$ is downstream of $\{S_b, G\}$.
$H(S_b, G) = H(T_a, S_b, G)$.

Ex 3: $T_c$ is downstream of $\{F, G\}$.
$H(F, G) = H(T_c, F, G)$. 
Proof: Max Rate $= \frac{2}{3}$
\[ H(S_a, F) + H(S_b, G) = H(S_a, T_b, F) + H(T_a, S_b, G) \]
sources = sinks
submodularity
downstreamness
sources = sinks
5 \log |\Gamma|
Max Rate = $2/3$

$\geq 5 \log |\Gamma|$
Max Rate $= \frac{2}{3}$

\[ \geq 5 \log |\Gamma| \]
Max Rate = $\frac{2}{3}$

$\geq 5 \log |\Gamma|$
Max Rate = 2/3

\[ \log |\Gamma| + \log |\Gamma| + \log |\Gamma| + \geq 5 \log |\Gamma| \]
Max Rate = 2/3

\[ \geq 3 \log |\Gamma| \]
Max Rate $= \frac{2}{3}$

\[ 2 \log |\Sigma| \geq 3 \log |\Gamma| \]
What is the Maximum Rate?

- Simple cut-based characterizations of max rate unsatisfactory.
  - Sparsity is wrong for directed graphs.
  - Meagerness is a loose upper bound.
- Do the entropy conditions give a tight upper bound on rate?
  - Unknown in general.
  - Many inequalities and many ways to combine; get giant LP.
Further Results: Coding in Undirected Graphs

- How do we even model this?
  - Rule out cyclic dependencies between edge functions.
  - Edge capacity bounds information flow in two directions.
- Entropy conditions carry over, e.g. downstreamness.
- Sparsity is a loose upper bound on rate.

Conjecture: In an undirected graph, the maximum multicommodity flow = the maximum network coding rate.

- We prove for an infinite class of “interesting” graphs.
Okamura-Seymour Example

- 4 commodities.
- Each edge has capacity 1.
- Sparsity 1.
- Maximum rate with network coding is also 3/4!
Okamura-Seymour Example

- Add new sources and sinks and the corresponding edges.
- Each source transmits one symbol from $\Gamma$.
- Each edge transmits one symbol from $\Sigma$.
- Want to show $\frac{\log |\Gamma|}{\log |\Sigma|} \leq \frac{3}{4}$.
- Use three different edge-cuts.
$H(S_b, U) = H(T_a, S_b, U)$
Okamura-Seymour Example - Cut #1

\[ H(S_b, U) = H(T_a, S_b, U) \]
\[ = H(S_a, S_b, U) \]
\[ = H(S_a, S_b, T_c, T_d, U) \]
\[ H(S_b, U) = H(T_a, S_b, U) = H(S_a, S_b, U) = H(S_a, S_b, T_c, T_d, U) = H(S_a, S_b, S_c, S_d) \]
\[ H(S_b, U) = H(T_a, S_b, U) = H(S_a, S_b, U) = H(S_a, S_b, T_c, T_d, U) = H(S_a, S_b, S_c, S_d) \]
Okamura-Seymour Example - Cut #1

\[ H(S_b, U) = H(T_a, S_b, U) \]
\[ = H(S_a, S_b, U) \]
\[ = H(S_a, S_b, T_c, T_d, U) \]
\[ = H(S_a, S_b, S_c, S_d) \]

\[ H(S_b) + H(U) \geq 4 \log |\Gamma| \]
\[ H(U) \geq 3 \log |\Gamma| \]
Okamura-Seymour Example - Cut #1

\[ H(S_b, U) = H(T_a, S_b, U) = H(S_a, S_b, U) = H(S_a, S_b, T_c, T_d, U) = H(S_a, S_b, S_c, S_d) \]
\[ H(S_b) + H(U) \geq 4 \log |\Gamma| \]
\[ H(U) \geq 3 \log |\Gamma| \]
\[ \geq 9 \log |\Gamma| \]
Okamura-Seymour Example - Cut #1

\[ H(S_b, U) = H(T_a, S_b, U) \]
\[ = H(S_a, S_b, U) \]
\[ = H(S_a, S_b, T_c, T_d, U) \]
\[ = H(S_a, S_b, S_c, S_d) \]

\[ H(S_b) + H(U) \geq 4 \log |\Gamma| \]
\[ H(U) \geq 3 \log |\Gamma| \]

\[ \geq 9 \log |\Gamma| \]
$H(S_a, V) = H(S_a, T_c, T_d, V)$
Okamura-Seymour Example - Cut #2

\[
H(S_a, V) = H(S_a, T_c, T_d, V) = H(S_a, T_b, S_c, S_d, V)
\]
Okamura-Seymour Example - Cut #2

\[ H(S_a, V) = H(S_a, T_c, T_d, V) \]
\[ = H(S_a, T_b, S_c, S_d, V) \]
\[ = H(S_a, S_b, S_c, S_d) \]
\[ H(V) \geq 3 \log |\Gamma| \]
Okamura-Seymour Example - Cut #2

\[ H(S_a, V) = H(S_a, T_c, T_d, V) \]
\[ = H(S_a, T_b, S_c, S_d, V) \]
\[ = H(S_a, S_b, S_c, S_d) \]

\[ H(V) \geq 3 \log |\Gamma| \]

\[ \geq 9 \log |\Gamma| \]
Okamura-Seymour Example - Cut #2

\[ H(S_a, V) = H(S_a, T_c, T_d, V) \]
\[ = H(S_a, T_b, S_c, S_d, V) \]
\[ = H(S_a, S_b, S_c, S_d) \]
\[ H(V) \geq 3 \log |\Gamma| \]

\[ \geq 9 \log |\Gamma| \]
Okamura-Seymour Example - Cut #3
Okamura-Seymour Example - Cut #3

\[ H(S_c, S_d, W) = H(T_b, S_c, S_d, W) \]
\[ = H(T_a, S_b, S_c, S_d, W) \]
\[ = H(S_a, S_b, S_c, S_d) \]
\[ H(W) \geq 2 \log |\Gamma| \]
Okamura-Seymour Example - Cut #3

\[ H(S_c, S_d, W) = H(T_b, S_c, S_d, W) \]
\[ = H(T_a, S_b, S_c, S_d, W) \]
\[ = H(S_a, S_b, S_c, S_d) \]
\[ H(W) \geq 2 \log |\Gamma| \]
\[ \geq 6 \log |\Gamma| \]
Okamura-Seymour Example - Cut #3

\[ H(S_c, S_d, W) = H(T_b, S_c, S_d, W) \]
\[ = H(T_a, S_b, S_c, S_d, W) \]
\[ = H(S_a, S_b, S_c, S_d) \]
\[ H(W) \geq 2 \log |\Gamma| \]

\[ \geq 6 \log |\Gamma| \]
Putting It Together

\[ 3(6 \log |\Sigma|) \geq 9 \log |\Gamma| + 9 \log |\Gamma| + 6 \log |\Gamma| \]

\[ 18 \log |\Sigma| \geq 24 \log |\Gamma| \]

\[ \frac{3}{4} \geq \frac{\log |\Gamma|}{\log |\Sigma|} \]
Network Coding vs. Multicommodity Flow

- Only comparable when each commodity has a single source and single sink.
- For this example, shown:
  \[
  \text{max flow rate} = \text{max network coding rate}
  \]
- Open: Is this true for all undirected graphs?
Additional Results

- Can prove the conjecture for all instances defined on bipartite graphs such that
  - Length 1 for all edges is dual optimal.
  - Distance between each source and sink is 2.
- Operational downstreamness: A set of edges $U$ is operationally downstream of a set $V$ if for all network coding solutions there exists a function mapping the symbols transmitted on edges in $V$ to edges in $U$.
  - In undirected graphs, we have a graph theoretic condition that characterizes operational downstreamness.
  - In directed graphs, the graph theoretic condition implies operational downstreamness.
Summary

• Capacity of information networks is poorly understood.
• Model for multicast is not appropriate for more general problems.
• Introduce a notion of rate.
• What is the maximum rate?
  ◦ Directed graphs: meagerness is a loose upper bound.
  ◦ Undirected graphs: sparsity is a loose upper bound.
• Introduced entropy relationships based on graph structure.
  ◦ Do these exactly characterize the rate?
Related Work

• By Monday, details will be available at:
  http:\\theory.csail.mit.edu/~arasala/thesis.pdf

• Song, Yeung and Cai ’03
  ○ For directed acyclic graphs, used similar entropy constraints to characterize an outer-bound on the feasible rate region.

• Jain et al. ’05
  ○ Developed similar entropy constraints for the general problem.
  ○ Independently derived same results for undirected graphs.
Can you solve this?
Length 1 is dual optimal
max flow = 8/15
Sparsity = 5/8