Network Routing Capacity

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Detailed results found in:

- R. Dougherty, C. Freiling, and K. Zeger
  “Linearity and Solvability in Multicast Networks”
  *IEEE Transactions on Information Theory*

- R. Dougherty, C. Freiling, and K. Zeger
  “Insufficiency of Linear Coding in Network Information Flow”
  *IEEE Transactions on Information Theory*
  (submitted February 27, 2004, revised January 6, 2005).

- J. Cannons, R. Dougherty, C. Freiling, and K. Zeger
  “Network Routing Capacity”
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  (submitted October 16, 2004).

Manuscripts on-line at: code.ucsd.edu/zeger
Definitions

- An **alphabet** is a finite set.

- A **network** is a finite d.a.g. with source messages from a fixed alphabet and message demands at sink nodes.

- A network is **degenerate** if some source message cannot reach some sink demanding it.
Definitions - scalar coding

- Each edge in a network carries an alphabet symbol.
- An edge function maps in-edge symbols to an out-edge symbol.
- A decoding function maps in-edge symbols at a sink to a message.
- A solution for a given alphabet is an assignment of edge functions and decoding functions such that all sink demands are satisfied.
- A network is solvable if it has a solution for some alphabet.
- A solution is a routing solution if the output of every edge function equals a particular one of its inputs.
- A solution is a linear solution if the output of every edge function is a linear combination of its inputs (typically, finite-field alphabets are assumed).
Definitions - vector coding

- Each edge in a network carries a vector of alphabet symbols.

- An edge function maps in-edge vectors to an out-edge vector.

- A decoding function maps in-edge vectors at a sink to a message.

- A network is vector solvable if it has a solution for some alphabet and some vector dimension.

- A solution is a vector routing solution if every edge function’s output components are copied from (fixed) input components.

- A vector linear solution has edge functions which are linear combinations of vectors carried on in-edges to a node, where the coefficients are matrices.

- A vector routing solution is reducible if it has at least one component of an edge function which, when removed, still yields a vector routing solution.
Definitions - \((k, n)\) fractional coding

- Messages are vectors of dimension \(k\).
  Each edge in a network carries a vector of at most \(n\) alphabet symbols.

- A \((k, n)\) fractional linear solution has edge functions which are linear combinations of vectors carried on in-edges to a node, where the coefficients are rectangular matrices.

- A \((k, n)\) fractional solution is a fractional routing solution if every edge function’s output components are copied from (fixed) input components.

- A \((k, n)\) fractional routing solution is minimal if it is not reducible and if no \((k, n')\) fractional routing solution exists for any \(n' < n\).
Definitions - capacity

- The ratio $k/n$ in a $(k, n)$ fractional routing solution is called an *achievable routing rate* of the network.

- The *routing capacity* of a network is the quantity

  \[ \epsilon = \sup\{ \text{all achievable routing rates} \}. \]

- Note that if a network has a routing solution, then the routing capacity of the network is at least 1.
Some prior work

- Some solvable networks do not have routing solutions (AhCaLiYe 2000).
- Every solvable multicast network has a scalar linear solution over some sufficiently large finite field alphabet (LiYeCa 2003).
- If a network has a vector routing solution, then it does not necessarily have a scalar linear solution (MéEfHoKa 2003).
- For multicast networks, solvability over a particular alphabet does not imply scalar linear solvability over the same alphabet (RaLe, MéEfHoKa, Ri 2003, DoFrZe 2004).
- For non-multicast networks, solvability does not imply vector linear solvability (DoFrZe 2004).
- For some networks, the size of the alphabet needed for a solution can be significantly reduced using fractional coding (RaLe 2004).
Our results

- Routing capacity definition.
- Routing capacity of example networks.
- Routing capacity is always achievable.
- Routing capacity is always rational.
- Every positive rational number is the routing capacity of some solvable network.
- An algorithm for determining the routing capacity.
Some facts

• Solvable networks may or may not have routing solutions.

• Every non-degenerate network has a \((k, n)\) fractional routing solution for some \(k\) and \(n\) (e.g. take \(k = 1\) and \(n\) equal to the number of messages in the network).
Example of routing capacity

This network has a linear coding solution but no routing solution.

Each of the $2k$ message components must be carried on at least two of the edges $e_{1,2}, e_{1,3}, e_{4,5}$. Hence, $2(2k) \leq 3n$, and so $\epsilon \leq 3/4$.

Now, we will exhibit a $(3, 4)$ fractional routing solution...
Let $k = 3$ and $n = 4$.

This is a fractional routing solution.

Thus, $3/4$ is an achievable routing rate, so $\epsilon \geq 3/4$.

Therefore, the routing capacity is $\epsilon = 3/4$. 

Example of routing capacity continued...
Example of routing capacity

The only way to get \( x \) to \( n_6 \) is \( n_1 \rightarrow n_3 \rightarrow n_4 \rightarrow n_6 \).
The only way to get \( y \) to \( n_5 \) is \( n_2 \rightarrow n_3 \rightarrow n_4 \rightarrow n_5 \).

\( e_{3,4} \) must have enough capacity for both messages.

Hence, \( 2k \leq n \), so \( \epsilon \leq 1/2 \).
Example of routing capacity continued...

Let $k = 1$ and $n = 2$.
This is a fractional routing solution.
Thus, $1/2$ is an achievable routing rate, so $\epsilon \geq 1/2$.
Therefore, the routing capacity is $\epsilon = 1/2$. 
Example of routing capacity

This network is due to R. Koetter.

Each source must emit at least $2k$ components and the total capacity of each source’s two out-edges is $2n$. Thus, $2k \leq 2n$, yielding $\epsilon \leq 1$. 
Example of routing capacity continued...

Let $k = 2$ and $n = 2$.

This is a fractional routing solution (as given in MéEfHoKa, 2003).

Thus, $2/2$ is an achievable routing rate, so $\epsilon \geq 1$.

Therefore, the routing capacity is $\epsilon = 1$. 
Each node in the 3rd layer receives a unique set of \( I \) edges from the 2nd layer.

Every subset of \( I \) nodes in layer 2 must receive all \( mk \) message components from the source. Thus, each of the \( mk \) message components must appear at least \( N - (I - 1) \) times on the \( N \) out-edges of the source. Since the total number of symbols on the \( N \) source out-edges is \( Nn \), we must have \( mk(N - (I - 1)) \leq Nn \) or equivalently \( k/n \leq N/(m(N - I + 1)) \). Hence, \( \epsilon \leq N/(m(N - I + 1)) \).
Let $k = N$ and $n = m(N - I + 1)$

There is a fractional routing solution with these parameters
(the proof is somewhat involved and will be skipped here).

Therefore, $\frac{N}{m(N - I + 1)}$ is an achievable routing rate, so

$\epsilon \geq \frac{N}{m(N - I + 1)}$.

Therefore, the routing capacity is $\epsilon = \frac{N}{m(N - I + 1)}$. 
Some special cases of the network:

- $m = 5$, $N = 12$, $I = 8$ (AhRi 2004)
  No binary scalar linear solution exist. It has a non-linear binary scalar solution using a $(5, 12, 5)$ Nordstrom-Robinson error correcting code. We compute that the routing capacity is $\epsilon = 12/25$.

- $m = 2$, $N = p$, $I = 2$ (RaLe 2003)
  The network is solvable, if the alphabet size is at least equal to the square root of the number of sinks. We compute that the routing capacity is $\epsilon = p/(2(p - 1))$.

- $m = 2$, $N = I = 3$
  Illustrates that the network’s routing capacity can be greater than 1. We obtain $\epsilon = 3/2$. 
For each message $m$, a directed subgraph of $G$ is an **$m$-tree** if it has exactly one directed path from the source emitting $m$ to each destination node which demands $m$, and the subgraph is minimal with respect to this property (similar to directed Steiner trees).

Let $T_1, T_2, \ldots$ be all such $m$-trees of a network. e.g., this network has two $x$-trees and two $y$-trees:
Define the following index sets:

\[ A(m) = \{ i : T_i \text{ is an } m\text{-tree} \} \]

\[ B(e) = \{ i : T_i \text{ contains edge } e \} \].

Denote the total number of trees \( T_i \) by \( t \).

For a given network, we call the following 4 conditions the **network inequalities**:

\[
\sum_{i \in A(m)} d_i \geq 1 \quad (\forall m \in M)
\]

\[
\sum_{i \in B(e)} d_i \leq \rho \quad (\forall e \in E)
\]

\[ 0 \leq d_i \leq 1 \]

\[ 0 \leq \rho \leq t \]

where \( d_1, \ldots, d_t, \rho \) are real variables. If a solution \((d_1, \ldots, d_t, \rho)\) to the network inequalities has all rational components, then it is said to be a **rational solution**. 

\((kd_i \) represents the number of message components carried by \( T_i \).\)
**Lemma:** If a non-degenerate network has a minimal fractional routing solution with achievable routing rate $r > 0$, then the network inequalities have a rational solution with $\rho = 1/r$.

**Lemma:** If the network inequalities corresponding to a non-degenerate network have a rational solution with $\rho > 0$, then there exists a fractional routing solution with achievable routing rate $1/\rho$.

By formulating a linear programming problem, we obtain:

**Theorem:** The routing capacity of every non-degenerate network is achievable.

**Theorem:** The routing capacity of every network is rational.

**Theorem:** There exists an algorithm for determining the network routing capacity.

**Theorem:** For each rational $r > 0$ there exists a solvable network whose routing capacity is $r$. 
**Network Coding Capacity**

- The **coding capacity** is

  \[ \sup \left\{ k/n \in \mathbb{Q} : \exists (k, n) \text{ fractional coding solution} \right\}. \]

- routing capacity $\leq$ linear coding capacity $\leq$ coding capacity

- Routing capacity is independent of alphabet size.
  Linear coding capacity is not independent of alphabet size.

- **Theorem:** The coding capacity of a network is independent of the alphabet used.
The End.
Insufficiency of Linear Network Codes

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A linearly solvable network.
\[ e_{23,31} = M_1a + M_2b \]
\[ e_{24,32} = M_3a + M_4c \]
\[ e_{25,33} = M_5b + M_6c \]
\[ e_{15,19} = M_7a + M_8b + M_9c \]

\[ c = M_{10}(M_1a + M_2b) + M_{11}(M_7a + M_8b + M_9c) \]
\[ b = M_{12}(M_3a + M_4c) + M_{13}(M_7a + M_8b + M_9c) \]
\[ a = M_{14}(M_5b + M_6c) + M_{15}(M_7a + M_8b + M_9c) \]
Equating coefficients of $a$, $b$, $c$ in the previous equations gives

$$I = M_{11}M_9 = M_{13}M_8 = M_{15}M_7$$

$$M_{10}M_1 = -M_{11}M_7$$

$$M_{10}M_2 = -M_{11}M_8$$

$$M_{12}M_3 = -M_{13}M_7$$

$$M_{12}M_4 = -M_{13}M_9$$

$$M_{14}M_5 = -M_{15}M_8$$

$$M_{14}M_6 = -M_{15}M_9$$

$$M_{10}(M_1a + M_2b) + M_{11}(M_7a + M_8b) = 0$$

$$M_{12}(M_3a + M_4c) + M_{13}(M_7a + M_9c) = 0$$

$$M_{14}(M_5b + M_6c) + M_{15}(M_8b + M_9c) = 0$$
A network linearly solvable over odd-characteristic fields.

\[ c = ((a + c) + (b + c) - (a + b)) \cdot 2^{-1} \]
In characteristic 2:

\[ M_7a \rightarrow a \]
\[ M_8b \rightarrow b \]
\[ M_9c \rightarrow c \]

\[ M_7a + M_8b \rightarrow M_{1a} + M_{2b} \]
\[ M_7a + M_9c \rightarrow M_{3a} + M_{4c} \]
\[ M_8b + M_9c \rightarrow M_{5b} + M_{6c} \]
A network linearly solvable over fields of characteristic 2.

\[(a + c) = (a + b) + (b + c)\]
In characteristic 2:

\[ M_7a + M_8b, \]
\[ M_7a + M_9c, \]
\[ M'_7c + M'_8d, \]
\[ M'_7c + M'_9e, \]

\[ \rightarrow a, b, c, d, e \]
**Definitions**

- A \((k, n)\) _fractional linear solution_ over \(F\) uses linear edge functions and decoding functions, where each source message is a vector of \(k\) elements of \(F\) and each edge carries a vector of \(n\) elements of \(F\).

- The _linear capacity_ of a network over \(F\) is the supremum of \(k/n\) over all pairs \((k, n)\) for which there exists a \((k, n)\) fractional linear solution over \(F\).

- A network is _asymptotically linearly solvable_ if its linear capacity is at least 1.
As shown before,

\[ I = M_{11}M_9 = M_{13}M_8 = M_{15}M_7 \]

\[ M_{10}M_1 = -M_{11}M_7 \]
\[ M_{10}M_2 = -M_{11}M_8 \]
\[ M_{12}M_3 = -M_{13}M_7 \]
\[ M_{12}M_4 = -M_{13}M_9 \]
\[ M_{14}M_5 = -M_{15}M_8 \]
\[ M_{14}M_6 = -M_{15}M_9. \]

Notice that:

\[ M_1, \ldots, M_9 \text{ are } n \times k \]
\[ M_{10}, \ldots, M_{15} \text{ are } k \times n \]
\[ M_7, M_8, M_9, M_{11}, M_{13}, M_{15} \text{ have rank } k \]
\[ M_{10}, M_{12}, M_{14} \text{ have rank at least } k - (n - k). \]
If a $k \times n$ matrix $M$ has rank at least $r$, then there is an $(n - r) \times n$ matrix $Q$ such that

$$\text{rank} \left( \begin{bmatrix} M \\ Q \end{bmatrix} \right) = n$$

and hence

$$Mx, Qx \rightarrow x.$$ 

For $M_{10}, M_{12}, M_{14}$, the corresponding matrices $Q_{10}, Q_{12}, Q_{14}$ are $2(n - k) \times n$. 
From

\[ M_{10}(M_1a + M_2b) = -M_{11}(M_7a + M_8b) \]

we get

\[ M_7a + M_8b, \quad Q_{10}(M_1a + M_2b) \rightarrow M_1a + M_2b. \]

Similarly,

\[ M_7a + M_9c, \quad Q_{12}(M_3a + M_4c) \rightarrow M_3a + M_4c \]
\[ M_8b + M_9c, \quad Q_{14}(M_5b + M_6c) \rightarrow M_5b + M_6c. \]

And we still have

\[ M_7a \rightarrow a, \quad M_8b \rightarrow b \quad M_9c \rightarrow c. \]
We now get in characteristic 2:

\[ M_7 a + M_8 b, \]
\[ M_7 a + M_9 c, \]
\[ Q_{10}(M_1 a + M_2 b), \]
\[ Q_{12}(M_3 a + M_4 c), \]
\[ Q_{14}(M_5 b + M_6 c), \]
\[ M_7' c + M_8' d, \]
\[ M_7' c + M_9' e, \]
\[ Q_{10}'(M_1' c + M_2' d), \]
\[ Q_{12}'(M_3' c + M_4' e), \]
\[ Q_{14}'(M_5' d + M_6' e) \]

\[ \rightarrow a, b, c, d, e. \]
From the previous page, in characteristic 2 we have:

\[ M_7a + M_8b, \]
\[ M_7a + M_9c, \]
\[ Q_{10}(M_1a + M_2b), \]
\[ Q_{12}(M_3a + M_4c), \]
\[ Q_{14}(M_5b + M_6c), \]
\[ M_7'c + M_8'd, \]
\[ M_7'c + M_9'e, \]
\[ Q_{10}'(M_1'c + M_2'd), \]
\[ Q_{12}'(M_3'c + M_4'e), \]
\[ Q_{14}'(M_5'd + M_6'e) \]
\[ \rightarrow a, b, c, d, e. \]

There are $5k$ independent components on the right, so there must be at least $5k$ components on the left. So,

\[ 4n + 6(2(n - k)) \geq 5k \]
\[ 16n \geq 17k \]
\[ 16/17 \geq k/n. \]
With substantial additional work, one can show that the complete example network has:

- linear capacity $4/5$ over odd-characteristic fields, and
- linear capacity $10/11$ over even-characteristic fields.

So the network is solvable, but not asymptotically linearly solvable.
Explicit counterexample network giving:

- Non-linear solution over 4-symbol alphabet.
- No vector linear solution for any dimension or any finite field.
- No $R$-linear solution over any $R$-module
  ($\therefore$ no linear solutions over Abelian groups or arbitrary rings for any dimension).
- Coding capacity is 1.
- Linear coding capacity over finite fields is $4/5$ or $10/11$ depending on parity of alphabet size.
- Linear codes are asymptotically insufficient over finite fields.
- Not solvable by means of convolutional coding or filter-bank coding.
Detailed results found in:

- R. Dougherty, C. Freiling, and K. Zeger
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The End.