Problems to Explore
Rutgers Young Scholars Program
Summer 2016

PROBLEM 1: THE AGES OF WOMEN?

Fifteen years ago a woman’s age was half her mother’s age. Fifteen years from now the woman’s age will be twice her daughter’s age. Right now the sum of the ages of the woman, her mother and her daughter is 100 years. How old are the woman, her mother and her daughter?

PROBLEM 2: A TRIP IN THE SLOW LANE

A man made a trip of 432 miles by automobile. If his average speed had been 6 miles per hour faster, the trip would have taken one hour less. How long did the trip take and what was his average speed?

PROBLEM 3: COOPERATIVE BRICKLAYING

A building contractor employs a master mason, a journeyman mason and an apprentice mason. The master mason and the journeyman working together can finish a brick wall in 10 hours. It takes the master and apprentice 12 hours to do the same job, while the journeyman and apprentice require 15 hours to complete the task.

If all three work together, how long does it take to build the wall?

PROBLEM 4: EVENS AND ODDS

In the multiplication shown at the right, the O’s represent odd digits and the E’s represent even digits. The O’s do not all represent the same digit, nor do the E’s. Also, as usual, the left-most digit in any number is not zero. Find all possibilities for the digits that give a multiplication with this pattern of even and odd numbers.

\[
\begin{array}{c}
O & O & E \\
O & E & E \\
E & O & E \\
E & E & O \\
\end{array}
\]
PROBLEM 5: ATTACKING QUEENS

Recall that in chess a queen attacks any square that is on a straight line — horizontally, vertically, or diagonally — from the square on which the queen stands. Regard the queen as also attacking the square which she occupies. What is the smallest number of queens required to attack all of the squares of a $6 \times 6$ chessboard?

PROBLEM 6: TANGENT CIRCLES

In the diagram to the right, the outer circle, centered at $C$, has diameter $AE$. The middle-sized circles, centered at $B$ and $D$, have respective diameters $AC$ and $CE$. The smaller circles, centered at $F$ and $G$, are just tangent to the outer circle and the circles centered at $B$ and $D$. If the outer circle has radius 1, what is the radius of the circle with center $F$?

PROBLEM 7: EASY MULTIPLICATIONS

The number $128205$ ends in 5 and has the property that multiplying the number by 4 can be accomplished by moving the ending 5 to the front of the number, or $4 \times 128205 = 512820$. Find a number that ends in 6 and has the property that multiplying the number by 4 can be accomplished by moving the ending 6 to the front of the number. Can you find a second such number?

PROBLEM 8: A FIBONACCI SUBSEQUENCE

The Fibonacci sequence $f_1, f_2, f_3, f_4, \ldots, f_n, \ldots$ is defined by $f_1 = 1, f_2 = 1$ and if $n \geq 1$, by the recursion relation $f_{n+2} = f_{n+1} + f_n$. In other words, from the third term onward, each term is the sum of the previous two terms.

The resulting sequence is: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, \ldots

A new sequence $g_1, g_2, g_3, g_4, \ldots, g_m, \ldots$ is formed by removing the 1st, 3rd, 5th, 7th, etc. terms from the Fibonacci sequence. The resulting sequence is: 1, 3, 8, 21, 55, 144, 377, \ldots. Since the terms are $f_2, f_4, f_6, f_8, \ldots, f_{2m}, \ldots$ it follows that $g_m = f_{2m}$. Find, if possible, constant numbers $A$ and $B$ so that if $m \geq 1$, the terms $g_m$ of the sequence can be calculated by the recursion relation: $g_{m+2} = Ag_{m+1} + Bg_m$. 
PROBLEM 9: TURNING PENNIES

Fourteen pennies are placed in a line with all heads up. Let $k$ be a positive integer. The objective is to turn over the pennies $k$ at a time, until all of the pennies have tails up. (Any individual penny may be turned over a number of times, but at the end it must be tails up.) In addition, this is to be accomplished using as few moves as possible. In your answer explain how you performed the task and why the number of moves you used was as small as possible.

(a) If $k = 3$, or in other words, the pennies are turned over three at a time.

(b) If $k = 4$.

(c) If $k = 5$.

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PROBLEM 10: COLORING A BRACELET

A six-bead bracelet is made out of red, yellow, and blue beads arranged so that neighboring beads have different colors. In how many different ways is this possible?

Note that for this problem two ways to color the bracelet are considered the same only when bead 1 is colored the same in both, bead 2 is colored the same in both, etc.

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PROBLEM 11: SUMS OF CONSECUTIVE ODD INTEGERS

There are two different ways of expressing the number 100 as a sum of consecutive positive odd integers:

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100 = 49 + 51 \\
100 = 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19
\]

What are the ways of expressing 1000 as a sum of consecutive positive odd integers?

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PROBLEM 12: CARPET CUTTING

A $7 \times 7$ rug has a $1 \times 1$ hole cut out of the center. Show how to cut it into two pieces that can be reassembled to form a $6 \times 8$ rug.
PROBLEM 13: SHUNTING LOCOMOTIVES

The diagram shows the assigned location of five locomotives in a railroad storage yard. Each number represents an engine, each circle represents a storage berth, and each line represents a track between two storage berths. The storage berth at the bottom is empty.

In the following diagrams, the locomotives were brought into the storage area and parked haphazardly with no regard for proper position. How can you use the empty storage berth and existing tracks to rearrange the engines into their assigned locations?

In (a), for example, your first step would be to move one of the engines 1, 3, or 5 to the empty bottom berth. If, for example, you moved 5 to this berth, you could next move 1, 4, or 5, to the berth vacated by 5. The last move, while not incorrect, is pointless, since you have simply returned to the original position and started over.

PROBLEM 14: CHECKERS

What is the largest number of checkers that can be placed on a $6 \times 6$ checkerboard so that no three consecutive checkers lie on a line – horizontal, vertical, or diagonal?

For example, pictured below are three placements of checkers on a $4 \times 4$ board. In each case there are nine checkers on the board, but the first two cases do not satisfy the condition, since there are three consecutive checkers on a vertical line in the first case and on a diagonal line in the second case. In the third case, the condition holds since no three checkers are consecutive on a line.

Show that nine is the largest number of checkers that can be placed on a $4 \times 4$ board subject to the above condition. Perform a similar analysis for a $6 \times 6$ board, and try to explain why the number of checkers that you placed is as large as possible.
PROBLEM 15: VISITING ALL LOCATIONS

Below are several examples of graphs. The circles are called vertices, and the horizontal and vertical roads connecting the vertices are called edges. In the graphs below, the edges have been colored, some white and some black. In this problem, the objective is to begin at some vertex and follow a route, along the edges, that passes through every vertex exactly once and returns to the starting vertex. In addition, the edges of the route must alternate in color: black, white, black, white, etc.

For the $4 \times 4$ grid graph below at the left, such a route is shown on the right. The route has been indicated by removing all edges not in the route. Start at any vertex and follow the included edges.

How many similar routes can you find for the $6 \times 6$ grid graph below?