About Mutations of Laurent Polynomials

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Motivation

Maximally Mutable Laurent Polynomials, Mirror Symmetry

Our project builds on a paper by Coates et al. titled *Maximally Mutable Laurent Polynomials* (2021).

The authors introduce a class of Laurent polynomials (polynomials in $\mathbb{C}[x_1^{\pm 1}, \ldots, x_i^{\pm 1}]$) which correspond under mirror symmetry to certain two-dimensional and three-dimensional Fano manifolds called **Maximally Mutable Laurent Polynomials**.

Project Statement

We are interested in the Laurent polynomial corresponding to projective three-space $\mathbb{P}^3$ under mirror symmetry given by:

$$P = x + y + z + \frac{1}{xyz}$$

In particular, we will study the graph formed by certain 'mutations' of P preserving an invariant known as its period integral.

Mutations in Three Variables

Definition

Let $N = \mathbb{Z}[x, y, z]$ be a lattice and let w be a primitive vector in the dual lattice. Let $a \in \mathbb{C}[\, w^{\perp} \cap N \,]$ be a Laurent polynomial. The pair (w, a) defines an automorphism of $\mathbb{C}(N)$ given by

$$\mu_{w,a} : \mathbb{C}(N) \longrightarrow \mathbb{C}(N), \quad x^j y^k z^\ell \longmapsto x^j y^k z^\ell \, a^{\, w(j,k,\ell)}.$$

If $\mu_{w,a}(f) = g \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, we call g a **mutation** of f.

The mutation graph of P is a graph whose vertices are all Laurent polynomials produced in this way, and edges are the mutations $\mu_{w,a}$.

Project Goals

- It is known that the mutation graph of the Laurent polynomial corresponding to projective two-space $\mathbb{P}^2$ is isomorphic to the Markov tree. Is there an embedding of the Markov tree in the mutation graph for $\mathbb{P}^3$?
- There is a correspondence between mutations of Laurent polynomials and transformations of their Newton polytopes. How can this be employed to study the mutation graph?
- Is the graph a tree? Simply connected? If not, what length cycles are there?

Building a Program

Computing mutations quickly became impossible by hand. I needed a program to efficiently compute and visualize mutations.

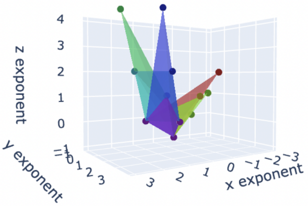
x + y + 2 y z / x + y z^2 / x^2 + 1 / x y z
a=1 + x^-1*z, w=(-1, 2, -1)
on-edge vertices: 5

x + y + y z / x + 1 / x y z + 1 / x^2 y
a=1 + x^-1*z, w=(-1, 1, -1)
on-edge vertices: 5

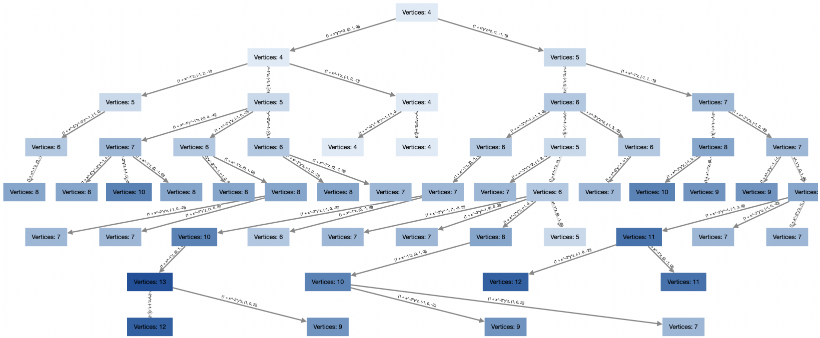
x^3 y^2 z^4 + 2 x^2 y z^2 + x + y + 1 / x y z
a=x*y*z^2 + 1, w=(2, 0, -1)
on-edge vertices: 5

x^2 y z^2 + x y^2 z^2 + x + y + 1 / x y z
a=x*y*z^2 + 1, w=(1, 1, -1)
on-edge vertices: 5

x + y + z + 1 / x y z
a=1 + x^-1*z, w=(1, -2, 1)
on-edge vertices: 4



Early Mutation Graph



The Markov Phenomenon

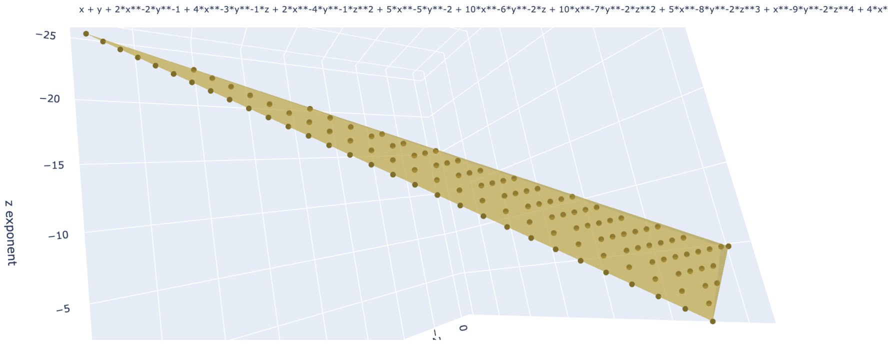
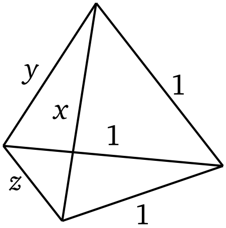


Figure: (2,5,29) Markov Triple

The Markov Phenomenon

Definition

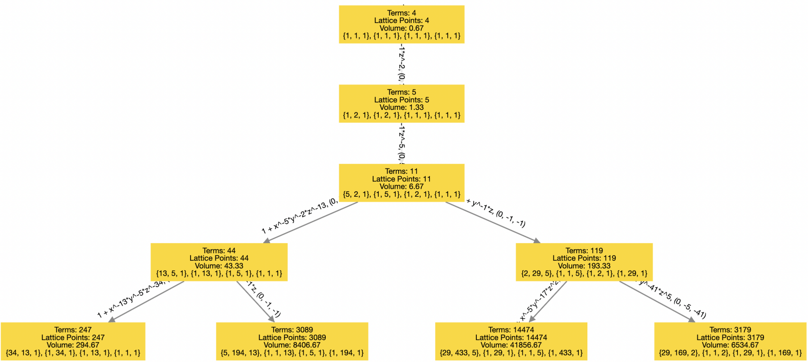
Let (x, y, z) be a Markov triple. A **Markov polytope** is a polytope with the following lattice lengths:



Conjecture

Let P be a Markov polytope. Then there exists a mutation $\mu_{w,a}$ which maps P to a Markov polytope Q with lengths $(x, y, 3xy - z)$.

The Markov Phenomenon



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